J. Fluid Mech. (1986), *vol.* 162, *pp.* 365–388
Printed in Great Britain

Chaotic motions in a weakly nonlinear model for surface waves

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(Received 3 April 1985)

Using the averaging method and a perturbation technique originally due to Melnikov (1963), we show that an N-degree-of-freedom model of weakly nonlinear surface waves due to Miles (1976) has transverse homoclinic orbits. This implies that Smale horseshoes, and hence sets of chaotic orbits, exist in the phase space. In this particular example, an irregular 'sloshing' of energy between two modes of oscillation results. We briefly discuss the relevance of our results to recent experimental work on parametrically excited surface waves.

1. Introduction: surface waves in a vertically oscillating container

When a container of fluid is subject to vertical periodic oscillations, a pattern of standing waves often appears on the free surface. Faraday (1831) and Lord Rayleigh (1883*a*, *b*) studied such 'crispations' experimentally and noted that the surface-oscillation frequency was typically one half that of the forcing frequency. Benjamin & Ursell (1954) showed that the linearized theory of irrotational motions of an ideal fluid in such a container led to an infinite set of Mathieu equations, and thus accounted for the observed frequency demultiplication. Moreover, while their theory was inviscid, the stability boundaries computed for the primary ($\frac{1}{2}$) resonance agreed well with experimental measurements.

More recently, Keolian *et al.* (1981), Keolian & Rudnick (1984), Gollub & Meyer (1983) and Ciliberto & Gollub (1984, 1985) have repeated the experiments with a view to detecting irregular or chaotic motions such as those found in many other fluid-mechanical systems (cf. Swinney & Gollub 1981). The latter authors, working with water, used a circular basin and studied the surface waveforms both spatially and temporally using surface-imaging and laser-beam-deflection methods. Gollub & Meyer (1983) examined the loss of stability of a single circularly symmetric mode and period-doubling bifurcations associated with it. At relatively high amplitudes they found temporally chaotic motions involving azimuthal spatial modulation. Ciliberto & Gollub (1984, 1985) studied the interaction of a pair of non-circularly symmetric modes at somewhat lower forcing amplitudes and found quasi-periodic and 'chaotically modulated' motions unaccompanied by significant increase in spatial complexity (their spatial Fourier transforms indicate that most of the energy continues to reside in the two 'linear' modes). Keolian *et al.* (1981) and Keolian & Rudnick (1984), using liquid helium and water in thin annular troughs, observed both period doubling and quasi-periodic motions apparently involving three modes, but their spatial data are less clear.

Somewhat earlier, Miles (1976) proposed a weakly nonlinear model for surface waves in such containers, which takes the form of an N-degrees-of-freedom Hamiltonian system with weak dissipation, the latter being primarily due to friction at the container walls (Miles 1967; Benjamin & Ursell 1954). Miles (1984*a–e*) has recently returned to ($N \geqslant 2$)-degree-of-freedom versions of this model in the light of recent developments in bifurcation and dynamical-systems theory. Concentrating on the case of horizontal excitations, he has found both steady (periodic) bifurcations and Hopf bifurcations to quasi-periodic motions and has numerically demonstrated that chaotic motions appear to be possible in the two-mode model, under various resonance conditions. However, on the basis of such numerical studies, he finds no evidence of chaos in a ($N > 2$)-mode model, at least for the moderate damping levels studied (Miles 1984*e*).

In this paper we examine Miles' model using the global perturbation techniques for 'almost Hamiltonian' systems developed by Holmes & Marsden (1982*a, b*, 1983) based on the Melnikov (1963) method. Our main aim is to outline and apply a method which enables one to rigorously prove the existence of chaotic motions in a wide class of such systems. The method has already been used in structural-vibration problems and has been applied to a nonlinear partial integrodifferential equation arising in beam theory (Holmes & Marsden 1981). For background information see the papers cited and Greenspan & Holmes (1983) and Guckenheimer & Holmes (1983, chap. 4). The present application introduces several new technical problems. Using the completely integrable limit obtained after averaging and in the absence of external forcing as a basis (Miles 1976, §7), we are able to prove that, for suitably small forcing and dissipation, chaotic motions of a precise type (Smale horseshoes, see Smale 1963, 1967) exist in both the two- and ($N > 2$)-degree-of-freedom models, for certain parameter ranges close to 2:1 subharmonic resonances. Our analysis is qualitative; we do not carry out calculations for specific container geometries (cf. Miles 1976, 1984*c–e*), but rather seek general results for a wide class of systems in the limit of weak nonlinearity, forcing and damping. In particular, the validity of our canonical transformation, averaging and perturbation procedure depends only on the existence of a 2:1 internal resonance and the absence of other internal resonances of the forms $\omega_k \pm \omega_l \pm \omega_m = 0$, where ω_k are the linearized natural frequencies defined in §2. Our results are certainly suitable for application to systems with anharmonically related normal modes, such as the circular cylinder of Miles (1976, 1984*c, d*). However, a rectangular tank (Miles 1984*e*), having approximately harmonically related modes, is more likely to exhibit additional resonances and our methods might require modification in that case. While we only consider the vertically forced container, we expect similar results to hold for horizontal forcing and for other resonance ratios. The chaotic motions we find, even when $N \gg 2$, involve primarily two surface modes, and are thus relatively simple spatially; the chaos is temporal.

Miles' model is derived using a finite-dimensional representation:

$$\eta(x,t) = \sum_{n-1}^{N} q_n(t)\,\psi_n(x); \quad \phi(x,y,t) = \sum_{n-1}^{N} \Phi_n(t)\,\chi_n(x,y), \tag{1.1}$$

for the surface $y = \eta(x,t)$ and velocity potential ϕ, for fluid of depth d at rest in a container of horizontal extent $D \subset \mathbb{R}^2$. Thus $y = -d$ and $x \in \partial D$ denote the fixed boundaries. Lagrangian and Hamiltonian formulations eventually lead to a finite-dimensional system of ordinary differential equations for the configuration variables $q_n(t)$ and their conjugate momenta $p_n(t)$. In the case of vertical excitation, the Hamiltonian is

$$H(q,p) = \tfrac{1}{2}\{\sum_{m,n} h_{mn}\,p_m\,p_n + \sum_{n} \bar{g}q_n^2\}, \tag{1.2}$$

where $\bar{g} = g + \ddot{X}_0(t)$ is the acceleration due to gravity and the imposed motion $X_0(t)$ of the container and $(\dot{\ })$ denotes d( )/dt; cf. Miles (1976, 1984*b*). We shall assume the container does not move horizontally. $[h_{mn}]$ is a symmetric matrix, the entries of which may be expanded as power series in the modal amplitudes q_n:

$$h_{mn} = \sum_{m} \delta_{mn} k_m + \sum_{l} h_{lmn} q_l + \tfrac{1}{2} \sum_{j,l} h_{jlmn} q_j q_l + \dots. \tag{1.3}$$

In (1.2)–(1.3) all sums are taken from 1 to N, the total number of modes (degrees of freedom) included. For details of the derivation of the model and computation of the coefficients k_n, h_{lmn}, etc., see Miles (1976).

An important requirement of our analysis is that the system be weakly dissipative, so that various high-order resonant effects are not present. We therefore include the generalized dissipative forces $F = \frac{1}{2}\Sigma_n f_n q_n^2$ in the Lagrangian (cf. Miles 1976, §3), which leads to the final system of $2N$ evolution equations

$$\left.\begin{aligned} \dot{q}_n &= k_n p_n + \sum_{l,m} h_{lmn} q_l p_m - \alpha_n q_n + \dots, \\ \dot{p}_n &= -(g + \ddot{X}_0(t)) q_n - \tfrac{1}{2} \sum_{l,m} h_{nlm} p_l p_m - \alpha_n p_n + \dots \end{aligned}\right\} \quad (n = 1, 2, \dots, N), \tag{1.4}$$

where $h_{ijk} = h_{ikj}$ and the vertical acceleration amplitude $|\ddot{X}_0(t)|$ and the damping coefficients α_n are supposed to be small, but not so small as the terms h_{jlmn} and higher in (1.3), which have been ignored. We note that, when dissipation and nonlinear terms are neglected, and $\ddot{X}_0(t)$ is taken to be sinusoidal, (1.4) reduces to the set of uncoupled Mathieu equations derived by Benjamin & Ursell (1954).

In this paper we start with the formulation (1.2)–(1.4) and assume that (perhaps after suitable scaling) the terms h_{lmn}, h_{jlmn}, etc. are small compared with the 'linear' terms k_n. In §2 we apply successive canonical transformations, averaging and truncation to obtain an integrable system in the case that $X_0(t) = 0$. This is essentially a repetition of Miles' (1976) analysis, with the vertical excitation term included. We find that the integrable system has homoclinic orbits (separatrix loops). In §3 we outline the Melnikov perturbation computations for the two-degree-of-freedom system and show that, for sufficiently small periodic excitations $X_0(t)$, transverse homoclinic orbits and consequently Smale horseshoes exist. In §4 we discuss technical problems which arise in the $N > 2$-degree-of-freedom forced Hamiltonian system and explain the role of damping in 'removing' the higher modes. However, the presence of damping introduces other complications into the analysis and the perturbation theory must be adapted to deal with these and to show that horseshoes can still occur. We relegate technical material and outlines of proofs of results in these two sections to the Appendix.

In §5 we describe the consequences of the existence of horseshoes for the full (unaveraged) system and the effects of higher-order terms. Finally, in §6 we describe the implications for surface motions of the fluid. We do not attempt to make comparisons between the predictions based on our qualitative analysis of Miles' weakly nonlinear model and the (strongly) nonlinear experiments of Keolian *et al.* (1981), Keolian & Rudnick (1984) and Gollub & Meyer (1983). However, our theory is more relevant to the Ciliberto–Gollub (1984, 1985) experiments and it is significant that in both cases the interaction of two modes leads to temporal chaos.

A philosophical note is in order here. Ruelle & Takens (1971) suggested that the chaotic orbits of strange attractors might provide a model for hydrodynamic turbulence. Strange attractors are complicated attracting sets of ordinary or partial differential equations or iterated mappings: the orbits within them exhibit sensitive dependence on initial conditions (Ruelle 1979) and they are typically of low dimension. Technically, a strange attractor is a closed subset, invariant under the flow of the equation, which attracts a neighbourhood of positive Lesbesgue measure of nearby orbits, and which contains a transverse homoclinic orbit and a dense orbit, cf. Guckenheimer & Holmes 1983, chapter 5.) Loosely speaking, the Ruelle–Takens suggestion implies that (certain types of) turbulence might involve only a few active degrees of freedom. Since then a great deal of work has been done with a view to demonstrating that strange attractors do occur in fluid mechanics. While several candidates have been identified in low-dimensional systems, including the famous Lorenz (1963) equations, which are a drastic truncation of a model for Rayleigh–Bénard convection, there is little theoretical evidence for strange attractors in realistic model equations for fluid problems. (The experimental evidence is somewhat better.) The present paper shows that some of the features of a low-dimension strange attractor occur in the surface-wave model, and thus provides more evidence, although it is not entirely satisfactory technically (dense orbits cannot be proven to exist; almost all orbits might be asymptotically periodic) or practically, since the temporally chaotic motion found is spatially simple.

We nonetheless feel that the present analysis is significant, since it provides a firm basis for statements about chaotic interactions between surface-wave modes. Moreover, in the analysis an important technical problem is unexpectedly illuminated. For $\epsilon = \mu = 0$ (cf. §2) the unperturbed system is linear and hence extremely degenerate. The Melnikov perturbation theory generally requires *strongly* nonlinear limits and 'large' unperturbed homoclinic orbits. Weak nonlinearities and the use of averaging typically lead to 'exponential smallness' and serious technical problems (Guckenheimer & Holmes 1983). Here, however, the three-way, resonant interaction between the two free modes and the forcing frequency preserves the mean force level as a symmetry-breaking perturbation even after averaging, and the difficulties are thereby avoided, see §§3 and 4. While the *single*-mode, averaged model considered by Miles (1984*b*, equation (4.1)) possesses homoclinic orbits similar to those of figures 1 and 2 (cf. figure 2 of Miles 1984*b*), for example, it appears that simple restoration of time-dependent terms removed by averaging leads to exponential-smallness problems.

After this paper was written, we learned of work in progress by I. Procaccia who is using invariant-manifold and normal form theory in an analysis of the partial differential equations of the irrotational, inviscid formulation of the surface-wave problem. He has computed numerically realistic values of the coefficients involved and preliminary results show a good comparison between theoretical stability boundaries and the measurements of Ciliberto & Gollub (1985).

2. Canonical transformations and averaging

We first make our smallness assumptions precise. Let $h_{ijk} = \epsilon \bar{h}_{ijk}$, $X_0(t) = \mu \bar{X}_0(t)$ and $\alpha_n = \mu \bar{\alpha}_n$ with $0 < \epsilon^2 \ll \mu \ll \epsilon \ll 1$ and assume h_{ijkl} and all higher-order terms in (1.3) are of $O(\epsilon^2)$. For example, one can take $\mu = \epsilon^{\frac{3}{2}}$. Next, letting $\omega_n = (gk_n)^{\frac{1}{2}}$ be the resonant frequency of the linearized nth mode and applying the canonical transformation

$$q_n = \left(\frac{2\omega_n \hat{p}_n}{g}\right)^{\frac{1}{2}} \cos \hat{q}_n, \quad p_n = -\left(\frac{2g\hat{p}_n}{\omega_n}\right)^{\frac{1}{2}} \sin \hat{q}_n \tag{2.1}$$

(Goldstein 1980), we obtain from (1.2) the transformed Hamiltonian

$$H = \sum_n \left\{\omega_n \hat{p}_n + \frac{\mu}{g} \ddot{X}_0(t)\, \hat{p}_n \cos^2 \hat{q}_n\right\}$$
$$+\epsilon \sum_{l,m,n} \bar{h}_{lmn} \left(\frac{2g\omega_l}{\omega_m \omega_n}\right)^{\frac{1}{2}} (\hat{p}_l \hat{p}_m \hat{p}_n)^{\frac{1}{2}} \cos \hat{q}_l \sin \hat{q}_m \sin \hat{q}_n + O(\epsilon^2); \quad (2.2)$$

cf. Miles (1976, equation (7.3)). Transformation of the dissipation terms

$$\mu(-\bar{\alpha}_n q_n, -\bar{\alpha}_n p_n)$$

for each mode leads to dissipation terms

$$\mu(0, -2\bar{\alpha}_n \hat{p}_n)$$

in the new coordinate system.

Our analysis now departs from Miles'. To proceed further, we take a specific $(2\pi/\omega)$-periodic sinusoidal forcing function

$$\ddot{X}_0(t) = \omega^2 \cos \omega t, \quad (2.3)$$

and we assume the resonance relationship $2\omega_1 \approx \omega_2 \approx \omega$ so that

$$2\omega_1 - \omega_2 = \epsilon\Omega; \quad \omega_2 - \omega = \epsilon\Delta; \quad \Omega, \Delta = O(1). \quad (2.4)$$

Moreover, we assume that the higher modes are non-resonant; specifically that $|\omega_k \pm \omega_l \pm \omega_m| \geqslant \delta \gg \epsilon$ for all $k, l, m \geqslant 3$, where δ is a fixed constant. Thus we are close to 2:1 internal resonance of the first two modes, coupled with 2:1 resonances and 1:1 resonances between the external force and the two free modes respectively. Other cases (such as $2\omega_1 \approx \omega_2 \approx \frac{1}{2}\omega$) can be treated similarly, although not all resonances lead to integrable, averaged systems amenable to the analysis developed here; cf. §6.

Using the generating function

$$\hat{p}_1 Q_1 + (\hat{p}_1 + 2\hat{p}_2)(Q_2 + \tfrac{1}{2}\omega t) + \sum_{k=3}^{N} \hat{p}_k (Q_k + \omega_k t), \quad (2.5)$$

and associated transformations

$$\left.\begin{aligned} \hat{q}_1 &= Q_1 + Q_2 + \tfrac{1}{2}\omega t, & \hat{p}_1 &= P_1, \\ \hat{q}_2 &= 2Q_2 + \omega t, & \hat{p}_2 &= \tfrac{1}{2}(P_2 - P_1), \\ \hat{q}_k &= Q_k + \omega_k t, & \hat{p}_k &= P_k \quad (k \geqslant 3), \end{aligned}\right\} \quad (2.6)$$

we obtain from (2.2) the transformed Hamiltonian

$$H = \epsilon\{\tfrac{1}{2}\Omega P_1 + \tfrac{1}{2}\Delta P_2 + \tfrac{1}{2}\beta P_1 (2(P_2 - P_1))^{\frac{1}{2}} \cos 2Q_1\} - \mu \frac{\omega^2 \omega_1}{4g} P_1 \cos(2Q_1 + 2Q_2)$$
$$+ \epsilon H^1(Q, P, \omega t) + \mu H^2(Q, P, \omega t) + O(\epsilon^2). \quad (2.7)$$

Here $\beta = (g/2\omega_2)^{\frac{1}{2}} [\bar{h}_{112} - (\omega_2/2\omega_1)\bar{h}_{211}] = O(1)$ and H^1 and H^2 are rapidly varying in t and have zero mean, regarded as functions of t; typical terms being $\cos(mQ_1 + nQ_2 + l\omega t)$. The dissipation terms transform to

$$\dot{P} = \mu(-2\bar{\alpha}_1 P_1, \ -2(\bar{\alpha}_1 - \bar{\alpha}_2) P_1 - 2\bar{\alpha}_2 P_2, \ -2\bar{\alpha}_3 P_3, \ldots, \ -2\bar{\alpha}_N P_N). \quad (2.8)$$

The perturbed Hamiltonian differential equation obtained from (2.7) is the form

$$\dot{x} = \epsilon X^0(x) + \mu X^1(x) + \epsilon Y(x, t) + \mu Z(x, t) + O(\epsilon^2), \quad (2.9)$$

where

$$x = \begin{pmatrix} P \\ Q \end{pmatrix} \in \mathbb{R}_+^N \times T^N$$

and Y, Z are $2\pi/\omega$ periodic in t. Applying the near-identity transformation

$$x = y + \epsilon U(y,t) + \mu V(y,t), \tag{2.10}$$

with $\partial U/\partial t = Y$, $\partial V/\partial t = Z$, (2.9) becomes

$$\dot{y} = \epsilon X^0(y) + \mu X^1(y) + O(\epsilon^2, \epsilon\mu, \mu^2), \tag{2.11}$$

where the $O(\epsilon^2, \epsilon\mu, \mu^2)$ remainder term in (2.11) is $2\pi/\omega$-periodic in t and uniformly bounded. The averaging theorem (Hale 1969; Guckenheimer & Holmes 1983) then implies that non-degenerate (i.e. hyperbolic) stationary and periodic solutions of the truncated system

$$\dot{y} = \epsilon X^0(y) + \mu X^1(y) \tag{2.12}$$

correspond respectively to $2\pi/\omega$-periodic and quasi-periodic solutions of the full system (2.11) or (2.9). More generally, if (2.12) possesses a hyperbolic invariant set Λ, then (2.9) will possess a hyperbolic invariant set Λ' near Λ, as we describe in §5 (Guckenheimer & Holmes 1983; Hirsch, Pugh & Shub 1977).

Carrying out averaging for the present Hamiltonian system with dissipation, (2.7)–(2.8), we obtain the *truncated* system

$$\left.\begin{aligned}
\dot{P}_1 &= \epsilon\beta P_1(2(P_2-P_1))^{\frac{1}{2}}\sin 2Q_1 - \mu\left\{\frac{\omega^2\omega_1}{2g}P_1\sin(2Q_1+2Q_2) + 2\alpha_1 P_1\right\},\\
\dot{Q}_1 &= \epsilon\left\{\tfrac{1}{2}\Omega + \frac{\beta(2P_2-3P_1)}{2(2(P_2-P_1))^{\frac{1}{2}}}\cos 2Q_1\right\} - \mu\left\{\frac{\omega^2\omega_1}{4g}\cos(2Q_1+2Q_2)\right\},\\
\dot{P}_2 &= -\mu\left\{\frac{\omega^2\omega_1}{2g}P_1\sin(2Q_1+2Q_2) + 2(\alpha_1-\alpha_2)P_1 + 2\alpha_2 P_2\right\},\\
\dot{Q}_2 &= \epsilon\left\{\tfrac{1}{2}\Delta + \frac{\beta P_1}{2(2(P_2-P_1))^{\frac{1}{2}}}\cos 2Q_1\right\},\\
\dot{P}_k &= -2\mu\alpha_k P_k,\\
\dot{Q}_k &= 0 \quad (k = 3, \dots, N).
\end{aligned}\right\} \tag{2.13}$$

Here we have dropped the bars on the $O(1)$ damping coefficients $\bar{\alpha}_n$. In Miles (1976, §7) the averaged system without forcing or damping is studied. Setting $\mu = 0$ in (2.13) we obtain a completely integrable, Hamiltonian system, since the angles $Q_2, \dots, Q_N$ are cyclic coordinates. The Hamiltonian energy

$$H = \epsilon\{\tfrac{1}{2}\Omega P_1 + \tfrac{1}{2}\Delta P_2 + \tfrac{1}{2}\beta P_1(2(P_2-P_1))^{\frac{1}{2}}\cos 2Q_1\}, \tag{2.14}$$

and the actions $P_2, \dots, P_N$ provide a set of N independent integrals in involution with respect to the Poisson bracket $\{F, G\} = \Sigma_{n-1}^N [(\partial F/\partial Q_n)(\partial G/\partial P_n) - (\partial F/\partial P_n)(\partial G/\partial Q_n)]$ (Goldstein 1980). We will use the integrable structure in the following sections.

We now describe this structure. Since in (2.13) $\dot{P}_k = 0$ for $k = 2, \dots, N$ when $\mu = 0$, all the P_k except P_1 remain constant and we need only study the (P_1, Q_1)-system. After integration of this system for fixed P_2, $Q_2(t)$ may be solved for directly. Standard phase-plane methods, including computation of fixed points and linearization, yield the phase portraits of figure 1 for P_2 held fixed and various values of the internal detuning parameter $\Omega = (2\omega_1 - \omega_2)/\epsilon$. The relevant region of the phase space is bounded by the line $P_1 = P_2$, on which the vector field is singular, so that the fixed

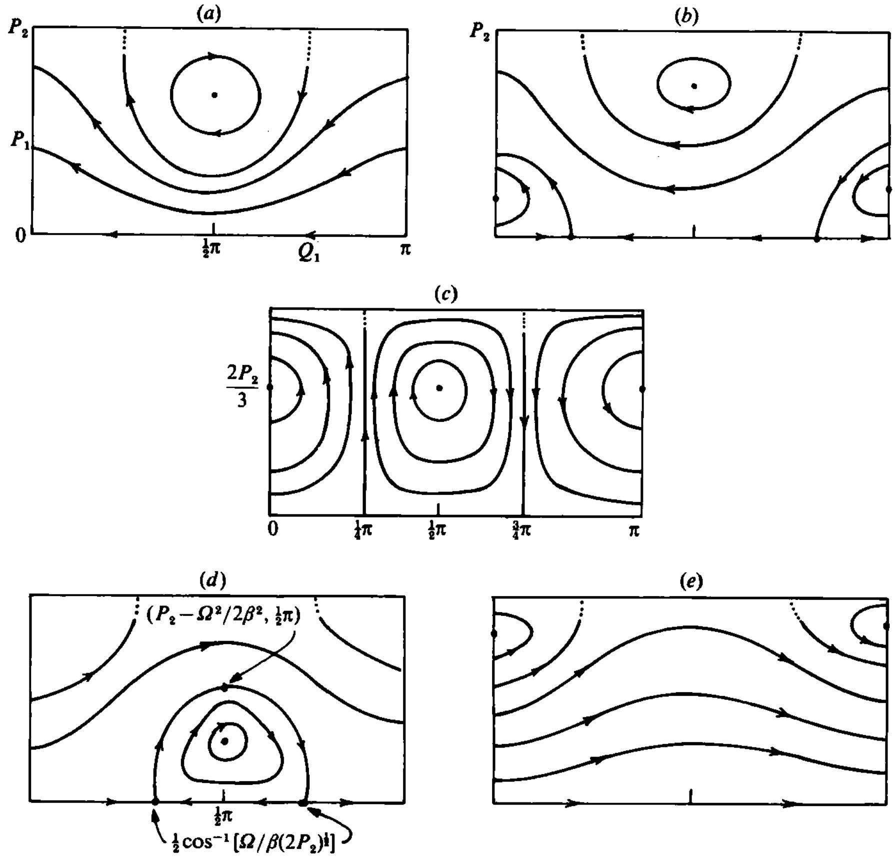

FIGURE 1. (P_1, Q_1)-phase-plane of (2.13) when $\mu = 0$, P_2 fixed. (*a*) $\Omega < -\beta(2P_2)^{\frac{1}{2}}$; (*b*) $-\beta(2P_2)^{\frac{1}{2}} < \Omega < 0$; (*c*) $\Omega = 0$; (*d*) $0 < \Omega < \beta(2P_2)^{\frac{1}{2}}$; (*e*) $\Omega > \beta(2P_2)^{\frac{1}{2}}$. The phase portrait is shown for $0 \leqslant Q_1 \leqslant \pi$ only, since it is π-periodic in Q_1.

points and orbits on this line have no physical meaning. For our purposes the heteroclinic orbits (saddle connections) to the fixed points at

$$(P_1, Q_1) = \left(0, \tfrac{1}{2}\cos^{-1}\left[\frac{-\Omega}{\beta(2P_2)^{\frac{1}{2}}}\right]\right)$$

and the closed orbits in their neighbourhoods are most important. Thus we will require that $|\Omega| < \beta(2P_2)^{\frac{1}{2}}$. The heteroclinic orbits lie on the level set

$$H = \epsilon\{\tfrac{1}{2}\Omega P_1 + \tfrac{1}{2}\Delta P_2 + \tfrac{1}{2}\beta P_1(2(P_2 - P_1))^{\frac{1}{2}}\cos 2Q_1\} = \tfrac{1}{2}\epsilon\Delta P_2, \tag{2.15}$$

or $P_1 = 0$ and $\Omega + \beta(2(P_2 - P_1))^{\frac{1}{2}}\cos 2Q_1 = 0$, and the heteroclinic orbits based at $P_1(0) = P_2 - \Omega^2/2\beta^2$ and $Q_1(0) = 0, \pi$ ($\Omega < 0$) or $Q_1(0) = \frac{1}{2}\pi, \frac{3}{2}\pi$ ($\Omega > 0$) are given by

$$P_1(t) = \left(P_2 - \frac{\Omega^2}{2\beta^2}\right)\operatorname{sech}^2(\epsilon(2\beta^2 P_2 - \Omega^2)^{\frac{1}{2}} t), \tag{2.16a}$$

$$Q_1(t) = \tfrac{1}{4}\epsilon\Omega\int_0^t \frac{P_1(s)}{P_2 - P_1(s)}\,ds \ (\text{resp.} +\pi, \tfrac{1}{2}\pi, \tfrac{3}{2}\pi), \tag{2.16b}$$

$$Q_2(t) = \tfrac{1}{4}\epsilon\int_0^t \frac{2\Delta P_2 - (\Omega + 2\Delta)P_1(s)}{P_2 - P_1(s)}\,ds + Q_2(0), \tag{2.16c}$$

as the reader can check by direct substitution into (2.13)–(2.15), with $\mu = 0$. (We note that $P_2 - P_1 > 0$ for physical relevance, so that singularities do not occur.) The closed orbits near this level set can be expressed in terms of elliptic functions. Henceforth we shall restrict our analysis to the case $\Omega > 0$ and the initial condition $Q_1(0) = \frac{1}{2}\pi$; the other cases are similar. Using (2.15), we can integrate (2.16*b, c*) easily to yield

$$2(Q_1(t) + Q_2(t)) = \epsilon \Delta t + \pi + 2Q_2(0), \tag{2.17}$$

which shows that an important special case occurs when $\Delta = 0$ (or $\Delta = O(\epsilon^2)$) and the external force is precisely in resonance with the second mode.

We remark that, while in the action-angle coordinates $(\hat{p}_i, \hat{q}_i)$ or (P_i, Q_i), the orbits described above appear to be *heteroclinic* rather than homoclinic, since they connect invariant closed curves at different Q_1 values, in the original (p_i, q_i)-coordinates, the surface $P_1 = \hat{p}_1 = 0$ corresponds to the origin in (p_1, q_1)-space and the orbits are *homoclinic* to the manifold $(q_1, p_1) = (0, 0)$; $\hat{p}_2 = \frac{1}{2}P_2 =$ constant. This will be important in §5.

In the next section we shall use (2.15)–(2.17) in perturbation calculations for the case $\mu \neq 0$.

3. The effects of weak forcing on the 2-mode truncated model

Melnikov (1963) suggested a regular perturbation method applicable to planar systems subject to small periodic perturbation. Holmes & Marsden (1982*a, b*, 1983) subsequently adapted this method to deal with two- and ($N > 2$)-degree-of-freedom Hamiltonian systems. We outline the method in the present context. Consider a two-degree-of-freedom system with Hamiltonian

$$H^\epsilon = H^0(q, p, I) + \epsilon H^1(q, p, I, \theta) + O(\epsilon^2) = \text{const.}, \tag{3.1}$$

so that, for $\epsilon = 0$, the angle θ is a cyclic coordinate and hence the action I is conserved. Thus I is a second integral, in addition to the total energy H^0. We assume that H^0 and I are integrals in involution. The perturbation theory addresses the question of how the 'completely integrable' structure breaks up for $\epsilon \neq 0$, small. Specifically, we assume that, for each I in some interval $L \subset \mathbb{R}$, the unperturbed (q, p) system,

$$\dot{q} = \frac{\partial H^0}{\partial p}, \quad \dot{p} = -\frac{\partial H^0}{\partial q}, \tag{3.2}$$

has a homoclinic orbit (= separatrix) $(q, p) = \bar{x}(t, I)$, with energy $H^0(\bar{x}, I)$, to a hyperbolic (= non-degenerate) saddle-point $\bar{x}_0(I)$, and that the unperturbed frequency

$$\Omega(q, p, I) \equiv \frac{\partial H^0}{\partial I}, \tag{3.3}$$

is strictly positive in a neighbourhood of every homoclinic orbit for $I \in L$. The method of *reduction* (Whittaker 1959, chap. 12), can then be applied to eliminate I by inverting (3.1):

$$I = \mathcal{L}^0(q, p; H^\epsilon) + \epsilon \mathcal{L}^1(q, p, \theta; H^\epsilon) + O(\epsilon^2), \tag{3.4}$$

and a simple calculation involving implicit differentiation of (3.1), with I given by (3.4), yields the reduced, 2π-periodically forced, one-parameter (H^ϵ-)family of single-degree-of-freedom systems:

$$q' = -\frac{\partial \mathcal{L}^0}{\partial p} - \epsilon \frac{\partial \mathcal{L}^1}{\partial p} + O(\epsilon^2), \quad p' = \frac{\partial \mathcal{L}^0}{\partial q} + \epsilon \frac{\partial \mathcal{L}^1}{\partial q} + O(\epsilon^2), \tag{3.5}$$

where ( )′ denotes $d/d\theta$. Also from (3.1)–(3.4) we obtain

$$\mathscr{L}^0 = [H^0(q,p)]^{-1}(H^\epsilon), \quad \mathscr{L}^1 = \frac{-H^1(q,p,\mathscr{L}^0(q,p;H^\epsilon),\theta)}{\Omega(q,p;\mathscr{L}^0(q,p;H^\epsilon))}. \tag{3.6}$$

See Holmes & Marsden (1982*a*, 1983) for computational details.

Our assumptions on H^0 imply that the unperturbed, reduced $\mathscr{L}^0$-systems ((3.5), $\epsilon = 0$) also have a family of homoclinic orbits $\bar{x}(\theta, I)$. Under the 2π-periodic forcing, these homoclinic orbits typically split, and if the *Melnikov function*

$$M(\theta_0) = \int_{-\infty}^{\infty} \{\mathscr{L}^0, \mathscr{L}^1\}(\bar{x}(\theta,I), I, \theta+\theta_0)\,d\theta, \quad I \in L, \tag{3.7}$$

has simple zeros, then, for $\epsilon \neq 0$ small, this splitting results in transverse homoclinic orbits to a hyperbolic, saddle-type, periodic orbit for each $I \in L$. Application of the Smale–Birkhoff homoclinic theorem (Smale 1963; Moser 1973; Guckenheimer & Holmes 1983) then reveals the existence of an invariant Cantor set for the Poincaré map on each constant-energy surface H^ϵ in some interval. This in turn implies the existence of chaotic motions and precludes the existence of analytic integrals of motion independent of the total energy H^ϵ. To avoid explicit computation of $\mathscr{L}^0$, $\mathscr{L}^1$ in examples, we have the identity

$$\int_{-\infty}^{\infty} \{\mathscr{L}^0, \mathscr{L}^1\}(\bar{x}(\theta,I), I, \theta+\theta_0)\,d\theta = \int_{-\infty}^{\infty} \left\{H^0, \frac{H^1}{\Omega}\right\}(\bar{x}(t,I), I, \bar{\theta}(t,I)+\theta_0)\,dt, \tag{3.8}$$

(Holmes & Marsden 1983). $\bar{\theta}(t,I)$ is defined in (3.12).

If we attempt to apply the theory directly to the averaged system (2.13), several problems arise. We shall tackle these in turn, and in doing so extend the Melnikov theory as necessary. We start by neglecting damping ($\alpha_k = 0$), so that $\dot{P}_k = \dot{Q}_k \equiv 0$ ($k \geqslant 3$), for the truncated system. Unfortunately we cannot conclude that the $(N-2)$-parameter family of invariant tori ($P_k = P_k(0)$, $Q_k = Q_k(0)$) persists, for the higher-order terms neglected in truncation typically destroy them. In particular, since they are all in resonance ($\dot{Q}_k \equiv 0$) we cannot appeal to KAM theory: Arnold (1978). In this section we therefore restrict ourselves to the two-degree-of-freedom model with Hamiltonian

$$H^\mu = \tfrac{1}{2}\epsilon\{\Omega P_1 + \Delta P_2 + \beta P_1(2(P_2-P_1))^{\frac{1}{2}}\cos 2Q_1\} - \frac{\mu\omega^2\omega_1}{4g} P_1 \cos(2Q_1+2Q_2). \tag{3.9}$$

(Here Q_1 and P_1 play the role of q, p, P_2 that of I and Q_2 of θ; also note that $0 \leqslant \mu \ll \epsilon$: μ/ϵ is the small parameter.)

We first note that, since

$$\Omega(Q_1, P_1; P_2) = \tfrac{1}{2}\epsilon\left\{\Delta + \frac{\beta P_1}{(2(P_2-P_1))^{\frac{1}{2}}}\cos 2Q_1\right\}, \tag{3.10}$$

and the homoclinic orbit $P_1(t) = (P_1 - (\Omega^2/2\beta^2))\,\mathrm{sech}^2\,(\epsilon(2\beta^2 P_1 - \Omega^2)^{\frac{1}{2}} t)$ lies on the level set $H^0 = \frac{1}{2}\epsilon\Delta P_2$, we require

$$|\Delta\Omega| > 2\beta^2 P_2 - \Omega^2 \quad (\Omega \neq 0), \tag{3.11}$$

for reduction to go through (cf. (2.16*c*)). Since passage through resonance, and the exact resonance $\Delta = 0$ itself, are interesting, we will develop a more general theory which does not appeal to reduction and hence does not require that $\Omega \neq 0$. J. Koiller (private communication) has already sketched a general theory along these lines, and in the Appendix we develop the special case of two degrees of freedom. We also deal with non-Hamiltonian perturbations, since we shall encounter these in analysing the damped model in §4. We remark that, if $\Omega \neq 0$ on the unperturbed homoclinic manifold, then the 'non-reduced' theory is equivalent to the reduced theory: see the Appendix. The key results are propositions (A 2), (A 3) and (A 4), which we apply below.

The Hamiltonian theory (proposition A 2) states that, if the Melnikov function

$$\left.\begin{aligned} M(I_0,\theta_0) &= \int_{-\infty}^{\infty} [\{H^0,H^1\}_{(q,p)}+\{H^0,\Omega\}_{(q,p)}\,I_1(t)]\,(\bar{x}(t,I_0),I_0,\bar{\theta}(t,I_0)+\theta_0)\,\mathrm{d}t,\\ \text{where}\qquad I_1(t) &= \int_0^t -\frac{\partial H^1}{\partial\theta}\,(\bar{x}(s,I_0),I_0,\bar{\theta}(s,I_0)+\theta_0)\,\mathrm{d}s\\ \text{and}\qquad \bar{\theta}(t,I_0) &= \int_0^t \frac{\partial H^0}{\partial I}\,(\bar{x}(s,I_0),I_0)\,\mathrm{d}s, \end{aligned}\right\} \tag{3.12}$$

has simple zeros as θ_0 varies for I_0 fixed in some interval $L \subset \mathbb{R}$, then there exists a one-parameter family Γ_ϵ of periodic orbits near the unperturbed orbits $(q,p,I,\theta) = (\bar{x}_0(I_0),I_0,\cdot\,)$ each of which possesses transverse homoclinic orbits. Here $\{H^0,H^1\}_{(q,p)}$ denotes the (q,p)-Poisson bracket $(\partial H^0/\partial q)\,(\partial H^1/\partial p)-(\partial H^0/\partial p)\,(\partial H^1/\partial q)$. Thus, as we describe in §5, the system has chaotic solutions.

Substituting the Hamiltonians and unperturbed solutions of the present problem into (3.12), recalling that here $\mu/\epsilon \ll 1$ plays the role of ϵ and that we assume $\epsilon^2 \ll \mu \ll \epsilon \ll 1$, we obtain

$$\left.\begin{aligned} \{H^0,H^1\}_{(P_1,Q_1)} &= \frac{\epsilon\omega^2\omega_1}{4g}\,P_1(t)\left\{\beta\left[[2(P_2-P_1(t)]^{\frac{1}{2}}\sin 2Q_2(t)\right.\right.\\ &\qquad\left.\left.-\frac{P_1(t)\cos 2Q_1(t)\sin(2Q_1(t)+2Q_2(t))}{[2(P_2-P_1(t))]^{\frac{1}{2}}}\right]+\Omega P_1(t)\sin(2Q_1(t)+2Q_2(t))\right\},\\ \{H^0,\Omega\}_{(P_1,Q_1)} &= \tfrac{1}{2}\epsilon^2\beta P_1(t)\left\{\frac{\beta(4P_2-3P_1(t))}{2(P_2-P_1(t))}\sin 2Q_1(t)\cos 2Q_1(t)+\frac{\Omega\sin 2Q_1(t)}{[2(P_2-P_1(t))]^{\frac{1}{2}}}\right\},\\ I_1(t) &= \int_0^t -\frac{\partial H^1}{\partial Q_2}\,\mathrm{d}s = \frac{-\omega^2\omega_1}{2g}\int_0^t P_1(s)\sin(2Q_1(s)+2Q_2(s))\,\mathrm{d}s. \end{aligned}\right\} \tag{3.13}$$

Using the expressions for $P_1(t)$, $2Q_1(t)$, $2Q_2(t)$ from (2.16)–(2.17), and noting that $P_1(t)$ is even, as are $\cos 2Q_1(t)$, $[2(P_2-P_1(t))]^{\frac{1}{2}}$ and $\cos\epsilon\Delta t$, while $\sin 2Q_1(t)$ and $\sin\epsilon\Delta t$ are odd, and noting that $F(t) = \int_0^t \text{even}\,(s)\,\mathrm{d}s$ is odd, we substitute (3.13) into (3.12) and finally obtain

$$M(I_0,\theta_0) = M(P_2,Q_{20}) = \frac{\epsilon\omega^2\omega_1}{4g}\,[\beta\mathscr{I}_1+\beta\mathscr{I}_2-\Omega\mathscr{I}_3+\epsilon\beta^2\mathscr{I}_4+\epsilon\beta\Omega\mathscr{I}_5]\sin 2Q_{20}, \tag{3.14}$$

where

$$\left.\begin{aligned} \mathscr{I}_1 &= \int_{-\infty}^{\infty} P_1(t)\,[2(P_2-P_1(t))]^{\frac{1}{2}}\cos 2Q_2(t)\,\mathrm{d}t,\\ \mathscr{I}_2 &= \int_{-\infty}^{\infty} \frac{P_1^2(t)\cos 2Q_1(t)\cos(\epsilon\Delta t)}{[2(P_2-P_1(t))]^{\frac{1}{2}}}\,\mathrm{d}t,\\ \mathscr{I}_3 &= \int_{-\infty}^{\infty} P_1(t)\cos(\epsilon\Delta t)\,\mathrm{d}t,\\ \mathscr{I}_4 &= \int_{-\infty}^{\infty}\left[\frac{P_1(t)\,(4P_2-3P_1(t))}{2(P_2-P_1(t))}\sin 2Q_1(t)\cos 2Q_1(t)\int_0^t P_1(s)\cos\epsilon\Delta s\,\mathrm{d}s\right]\mathrm{d}t,\\ \mathscr{I}_5 &= \int_{-\infty}^{\infty}\left[\frac{P_1(t)\sin 2Q_1(t)}{[2(P_2-P_1(t))]^{\frac{1}{2}}}\int_0^t P_1(s)\cos\epsilon\Delta s\,\mathrm{d}s\right]\mathrm{d}t. \end{aligned}\right\} \tag{3.15}$$

While the asymptotic behaviour of $P_1(t) = [P_2 - (\Omega^2/2\beta^2)]\,\mathrm{sech}^2\,[\epsilon(2\beta^2 P_2 - \Omega^2 t)^{\frac{1}{2}}]$ as $t \to \pm\infty$ guarantees that all the integrals converge, only $\mathscr{I}_3$ can be evaluated explicitly (by the method of residues). However, we note that, provided the quantity $\mathscr{I} = \beta(\mathscr{I}_1 + \mathscr{I}_2) - \Omega\mathscr{I}_3 + \epsilon\beta(\beta\mathscr{I}_4 + \Omega\mathscr{I}_5) \neq 0$, then $M(P_2, Q_{20})$ necessarily has simple zeros at $Q_{20} = \pm\frac{1}{2}n\pi$ for all n.

To show that parameter values exist for which the integral $\mathscr{I}$ is non-zero, we use (2.16)–(2.17) and rescale time ($\epsilon t = \tau, \epsilon s = \zeta$) to obtain

$$\frac{\epsilon\omega^2\omega_1}{4g}\mathscr{I} = \frac{\omega^2\omega_1}{4g}\left(P_2 - \frac{\Omega^2}{2\beta^2}\right)\left[\beta I_1 - \beta\left(P_2 - \frac{\Omega^2}{2\beta^2}\right) I_2 - \Omega I_3 + \beta\left(P_2 - \frac{\Omega^2}{2\beta^2}\right)(\beta I_4 - \Omega I_5)\right]\sin 2Q_{20}, \tag{3.16}$$

where

$$\left.\begin{aligned}
I_1 &= \int_{-\infty}^{\infty} S^2(\tau)\left(2P_2 - \left(2P_2 - \frac{\Omega^2}{\beta^2}\right)S^2(\tau)\right)^{\frac{1}{2}}\cos(C(\tau))\,\mathrm{d}\tau,\\
I_2 &= \int_{-\infty}^{\infty} \frac{S^4(\tau)\cos(D(\tau))\cos(\Delta\tau)}{\left(2P_2 - \left(2P_2 - \frac{\Omega^2}{\beta^2}\right)S^2(\tau)\right)^{\frac{1}{2}}}\,\mathrm{d}\tau,\\
I_3 &= \int_{-\infty}^{\infty} S^2(\tau)\cos(\Delta\tau)\,\mathrm{d}\tau,\\
I_4 &= \int_{-\infty}^{\infty} \frac{S^2(\tau)\left[4P_2 - 3\left(P_2 - \frac{\Omega^2}{2\beta^2}\right)S^2(\tau)\right]}{\left[2P_2 - \left(2P_2 - \frac{\Omega^2}{\beta^2}\right)S^2(\tau)\right]}\sin(D(\tau))\cos(D(\tau))\,E(\tau)\,\mathrm{d}\tau,\\
I_5 &= \int_{-\infty}^{\infty} \frac{S^2(\tau)\sin(D(\tau))}{\left(2P_2 - \left(2P_2 - \frac{\Omega^2}{\beta^2}\right)S^2(\tau)\right)^{\frac{1}{2}}}\,E(\tau)\,\mathrm{d}\tau;
\end{aligned}\right\} \tag{3.17}$$

and

$$\left.\begin{aligned}
S(\tau) &= \mathrm{sech}\,[(2\beta^2 P_2 - \Omega^2)^{\frac{1}{2}}\tau],\\
C(\tau) &= \frac{1}{2}\int_0^{\tau} \frac{2\Delta(1 - \Omega^2/2\beta^2 P_2)^{-1} - (\Omega + 2\Delta)S^2(\zeta)}{(1 - \Omega^2/2\beta^2 P_2)^{-1} - S^2(\zeta)}\,\mathrm{d}\zeta,\\
D(\tau) &= \tfrac{1}{2}\Omega\int_0^{\tau} \frac{S^2(\zeta)}{(1 - \Omega^2/2\beta^2 P_2)^{-1} - S^2(\zeta)}\,\mathrm{d}\zeta,\\
E(\tau) &= \int_0^{\tau} S^2(\zeta)\cos(\Delta\zeta)\,\mathrm{d}\zeta.
\end{aligned}\right\} \tag{3.18}$$

We note that this rescaling reveals that $M(P_2, Q_{20})$ does not depend on ϵ. Now when $\tau = 0$, the integrands of the I_i above are as follows: I_1: Ω/β; I_2: β/Ω; I_3: $+1$; while the integrands of I_4 and I_5 are both zero, since these integrals involve the products of odd functions. Thus we have the total integrand at $\tau = 0$ of:

$$-\frac{\beta^2\omega^2\omega_1}{4g\Omega}\left(P_2 - \frac{\Omega^2}{2\beta^2}\right)^2 \neq 0. \tag{3.19}$$

To guarantee that the non-zero contribution near $\tau = 0$ dominates the whole integral, it is merely necessary to ensure that the common factor $S(\tau)$ is 'steep' enough, so that the decay of this term as $|\tau|$ grows renders the contributions of I_4 and I_5 insignificant. This occurs if $(2\beta^2 P_2 - \Omega^2)^{\frac{1}{2}}$ is sufficiently large relative to $|\Delta|$, for then $E(\tau)$ is a relatively slowly varying function. Thus, applying (A 1) and (A 2) of the Appendix, we have proved:

THEOREM 3.1. *Suppose that* $(2\beta^2 P_2 - \Omega^2)$ *is positive and sufficiently large relative to* $|\Delta|$. *Then, for* $\mu/\epsilon \neq 0$ *sufficiently small, the two-degree-of-freedom Hamiltonian system* (3.9) *possesses a one-parameter family of invariant, closed curves close to the set* $(Q_1, P_1) = [\frac{1}{2}\cos^{-1}(-\Omega/\beta(2P_2)^{\frac{1}{2}}), 0]$; $P_2 =$ *constant. Each member of this set has transverse homoclinic orbits.*

Remark. Since the integrand (3.18) is non-zero for $|\Delta|$ small compared with $(2\beta^2 P_2 - \Omega^2)$, and since it is an analytic function of all the parameters, it has at most only a finite number of isolated zeros in any bounded region of parameter space. Thus for *almost all* parameter choices we can assert that $M(P_2, Q_{20})$ has simple zeros at $\frac{1}{2}n$, $n = 0, \pm 1, \pm 2, \ldots$.

We will discuss the implications of this result in §5, after we consider the more realistic model of (2.13), involving $N > 2$ modes and suitably weak damping.

4. The $(N > 2)$-mode truncated model subject to weak forcing and damping

While the undamped $(N > 2)$-mode *truncated* model of (2.13) is trivially soluble for $Q_k =$ const, $P_k =$ const $(k \geqslant 3)$, delicate resonance effects in the higher-order terms $(O(\epsilon^2), O(\epsilon\mu))$ which have been neglected, can be expected to destroy the family of invariant manifolds given by these solutions. Thus the inclusion of damping is essential. Fortunately, damping acts selectively, and straightforward integration of the $k \geqslant 3$ components of (2.13) shows the following:

PROPOSITION 4.1. *All solutions of the truncated system* (2.13) *approach the* $(N+2)$-*dimensional invariant manifold* $P_k = 0$, $k = 3, 4, \ldots, N$, *at an exponential rate as* $t \to \infty$.

This result implies that we can ignore behaviour in all modes other than the first two, for the attracting manifold $\{P_k = 0;\ k \geqslant 3\}$ is *normally hyperbolic* and higher-order terms cannot destroy it, although they may perturb it slightly (Hirsch *et al.* 1977, and cf. §5 below). Although we now have a two-degree-of-freedom system again, some problems absent from §3 intrude. In particular, the invariant closed curves (periodic orbits) $\{(Q_1, P_1) \approx [\frac{1}{2}\cos^{-1}(-\Omega/\beta(2P_2)^{\frac{1}{2}}), 0],\ P_2 \approx \text{const}\}$ of Theorem 3.1 all disappear when $\alpha_2 \neq 0$, as the P_2 evolution equation,

$$\dot{P}_2 = -\mu\left\{\frac{\omega^2\omega_1}{2g} P_1 \sin(2Q_1 + 2Q_2) + 2(\alpha_1 - \alpha_2) P_1 + 2\alpha_2 P_2\right\}, \tag{4.1}$$

clearly shows, for $P_2 = P_2(0)\, e^{-2\mu\alpha_2 t}$ when $P_1 = 0$. However, (4.1) suggests that orbits that approach $P_1 = 0$, but are not asymptotic to it, might balance dissipative effects due to α_1, α_2 with excitatory effects due to the Hamiltonian term $P_1 \sin(2Q_1 + 2Q_2)$, provided that $(2Q_1 + 2Q_2)$ varies sufficiently slowly (while P_1 is small) to remain of one sign, specifically negative, so as to supply energy to P_2. Before we apply proposition A 4 to check this, we compute the non-Hamiltonian Melnikov function of equation (A 13) and apply proposition A 3. The computation yields one term in addition to the Hamiltonian contribution of (3.17):

$$\int_{-\infty}^{\infty} \epsilon \left\{\tfrac{1}{2}\Omega + \frac{\beta(2P_2 - 3P_1(t))}{2(2P_2-2P_1(t))^{\frac{1}{2}}} \cos 2Q_1(t)\right\} 2\alpha_1 P_1(t)\,\mathrm{d}t$$

$$= \alpha_1 \left(P_2 - \frac{\Omega^2}{2\beta^2}\right) \int_{-\infty}^{\infty} S^2(\tau) \left[\Omega + \frac{\beta\left(2P_2 - 3\left(P_2 - \frac{\Omega^2}{2\beta^2}\right) S^2(\tau)\right)}{\left(2P_2 - \left(2P_2 - \frac{\Omega^2}{\beta^2}\right) S^2(\tau)\right)^{\frac{1}{2}}} \cos(D(\tau))\right] \mathrm{d}\tau. \quad (4.2)$$

(Recall that $S(\tau) = \operatorname{sech}(2\beta^2 P_2 - \Omega^2)^{\frac{1}{2}}\tau)$). Clearly, for α_1 sufficiently small compared with the common factor $\omega^2\omega_1/4g$ of the 'Hamiltonian' integral of (3.17), this additional Q_{20}-independent term cannot remove the simple zeros of $M(P_1, Q_{20})$. Thus the perturbed 2-manifolds near to $(Q_1, P_1) = (\frac{1}{2}\cos^{-1}(-\Omega/\beta(2P_2)^{\frac{1}{2}}), 0)$ still have transverse homoclinic orbits for small α_1. However, the presence of (4.2) in $M(P_2, Q_{20})$ (see (3.14)–(3.18)) does modify our estimate of the value of Q_{20} for which M has simple zeros. Specifically we find

$$-\frac{\omega^2\omega_1}{4g}(2\beta^2P_2 - \Omega^2)\,[A + B(2\beta^2P_2 - \Omega^2)] \sin 2Q_{20} + \alpha_1(2\beta^2P_2 - \Omega^2)\,C = 0,$$

or

$$\frac{\omega^2\omega_1}{4g} \sin 2Q_{20} = \frac{\alpha_1 C}{A + B(2\beta^2P_2 - \Omega^2)}, \quad (4.3)$$

where A, B and C involve the integrals of (3.17) and (4.2) and therefore also contain the expression $(2\beta^2P_2 - \Omega^2)$. We will use (4.3) below.

We now return to the question of recurrence. We compute the variation in P_2 around a segment of the unperturbed homoclinic manifold bounded away from $P_1 = 0$:

$$\left.\begin{aligned}
\Delta P_2 &= \int_{-T}^{T} \dot{P}_2(t)\,\mathrm{d}t \\
&= \int_{-T}^{T} -\mu\left[\frac{\omega^2\omega_1}{2g} P_1 \sin(2Q_1(t) + 2Q_2(t)) + 2(\alpha_1 - \alpha_2) P_1(t) + 2\alpha_2 P_2\right] \mathrm{d}t, \\
&= \frac{-\mu}{\epsilon}\left(P_2 - \frac{\Omega^2}{2\beta^2}\right) \int_{-\epsilon T}^{\epsilon T} S^2(\tau) \left[\frac{\omega^2\omega_1}{2g} \sin(\Delta\tau + \pi + 2Q_{20}) + 2(\alpha_1 - \alpha_2)\right] \mathrm{d}\tau - 4\mu\alpha_2 P_2 T, \\
&= \frac{\mu\omega^2\omega_1}{4\epsilon\beta^2 g}(2\beta^2P_2 - \Omega^2) \int_{-\epsilon T}^{\epsilon T} S^2(\tau) \cos\Delta\tau\,\mathrm{d}\tau \sin 2Q_{20} \\
&\quad - \frac{2\mu(\alpha_1 - \alpha_2)}{\epsilon\beta^2}(2\beta^2P_2 - \Omega^2)^{\frac{1}{2}} \tanh(\epsilon(2\beta^2P_2 - \Omega^2)^{\frac{1}{2}} T) - 4\mu\alpha_2 P_2 T.
\end{aligned}\right\} \quad (4.4)$$

The limits of this integral must be taken as $T = K\ln(\epsilon/\mu)$, as described in the Appendix (proposition A 4), for then we guarantee hyperbolicity in the (P_1, Q_1)-directions for the Poincaré return map. (Recall that, in our example, μ/ϵ plays the role of the small parameter in the general theory and that $\epsilon > 0$ is fixed, albeit also small.) The first integral of (4.3) cannot be evaluated explicitly, but since $T = K\ln(\epsilon/\mu)$ and $S^2(\epsilon T) \to 0$ (and $\tanh^2(\epsilon T \to 1)$) exponentially as $T \to \infty$, (4.3) can be evaluated with an error of $O((\mu/\epsilon)^{\epsilon K\sqrt{(2\beta^2P_2 - \Omega^2)}})$ to yield an expression of the form:

$$\Delta P_2 = \frac{\mu}{\epsilon\beta^2}\frac{\omega^2\omega_1\pi\Delta}{4g} \operatorname{cosech}\left[\frac{\Delta\pi}{2(2\beta^2P_2 - \Omega^2)^{\frac{1}{2}}}\right] \sin 2Q_{20}$$
$$- 2(\alpha_1 - \alpha_2)(2\beta^2P_2 - \Omega^2)^{\frac{1}{2}} - 4\epsilon\beta^2\alpha_2 P_2 K \ln\left(\frac{\epsilon}{\mu}\right) + \ldots \quad (4.5)$$

If $|\Delta| \ll (2\beta^2 P_2 - \Omega^2)^{\frac{1}{2}}$ (as in theorem 3.1), (4.4) simplifies to

$$\Delta P_2 = \frac{\mu}{\epsilon\beta^2}(2\beta^2 P_2 - \Omega^2)^{\frac{1}{2}}\left[\frac{\omega^2\omega_1}{2g}\sin 2Q_{20} - 2(\alpha_1 - \alpha_2)\right] - 4\epsilon\beta^2\alpha_2 P_2 K \ln\left(\frac{\epsilon}{\mu}\right) + \dots \quad (4.6)$$

Since $\epsilon^2 \ll \mu \ll \epsilon$, $\epsilon \ln(\epsilon/\mu)$ is small and zeros of ΔP_2 are obtained by a balance of the leading term. Thus, for $2\beta^2 P_2 - \Omega^2 > 0$ (cf. §2), we require

$$(\omega^2\omega_1/2g)\sin 2Q_{20} \approx 2(\alpha_1 - \alpha_2)$$

or, using (4.3),

$$\alpha_1 C \approx (\alpha_1 - \alpha_2)(A + B(2\beta^2 P_2 - \Omega^2)). \quad (4.7)$$

Since A, B, C themselves contain the expression $(2\beta^2 P_2 - \Omega^2)$ and cannot be evaluated explicitly, we cannot give precise conditions on the parameters $(\omega, \omega_1, \Omega, \Delta, \beta, \alpha_i)$ for which the hypothesis of propositions A 3 and A 4 are simultaneously satisfied. However, it should be clear that values of $\alpha_1, \alpha_2 > 0$ can be chosen so that (4.7) is satisfied with $P_2 > \Omega^2/2\beta^2$. Let us denote the resulting zero by P_2^*. Further slight variation of α_1, α_2 will guarantee that P_2^* is a simple zero $((\partial\Delta P_2/\partial P_2)(P_2 = P_2^*) \neq 0)$. Thus we can conclude:

Theorem 4.2. *For $0 < \epsilon^2 \ll \mu \ll \epsilon \ll 1$ open sets of parameter values Δ, Ω, α_i can be found for which the damped, truncated N-degree-of-freedom Hamiltonian system* (2.13) *induces a Poincaré map Φ on the cross-section $Q_2 = const.$, such that Φ has an invariant, hyperbolic Cantor set Λ near the set $(Q_1, P_1, P_2, P_k) = (\frac{1}{2}\cos^{-1}(-\Omega/\beta(2P_2^*)^{\frac{1}{2}}), 0, P_2^*, 0)$; $k = 3, \dots, N$. The dynamics of Φ restricted to Λ are conjugate to those of Smale's horseshoe, so that Φ contains infinitely many periodic orbits, including orbits of arbitrarily long period, along with bounded, non-periodic (= 'chaotic') orbits.*

We next turn to the implications of this result for the full (non-averaged) system. We will discuss the physical consequences of the periodic and chaotic orbits of the horseshoe for the surface motions of the fluid in §6.

5. Implications for the $(N > 2)$-mode full model

The conclusions of this section are based on the theory of normally hyperbolic manifolds due to Hirsch *et al.* (1977). The theory has already been used in §§3 and 4 and the Appendix to argue that certain invariant sets in the integrable, undamped $(\mu = 0)$, truncated system persist for $\mu \neq 0$. Briefly, if a system $\dot{x} = f(x)$ has a hyperbolic, invariant set Ω, then any nearby system $\dot{x} = f(x) + g(x)$ will have a similar (= topologically equivalent) set Ω', provided the perturbation is of sufficiently small size $\|g\|$ in the neighbourhood of Ω: that is, g and its derivatives are required to be uniformly small. Similar results apply to mappings $x \to F(x)$ and $x \to F(x) + G(x)$. The added complication in this paper is that the 'perturbed' system – the full, unaveraged system – is explicitly time dependent, whereas the averaged system is not. We will cope with this by suspending the time-dependent vector field.

The material above, culminating in Theorem 4.2, shows that the N-mode, damped, averaged, truncated system (2.13) induces a Poincaré return map Φ on the $(2N-1)$-dimensional cross-section $Q_2 = 0$, such that Φ has an invariant Cantor set Λ: a Smale horseshoe. This in turn implies that the flow of (2.13) contains a (topologically one-dimensional) bundle of solutions Ω whose cross-section is Λ. This set lies near the

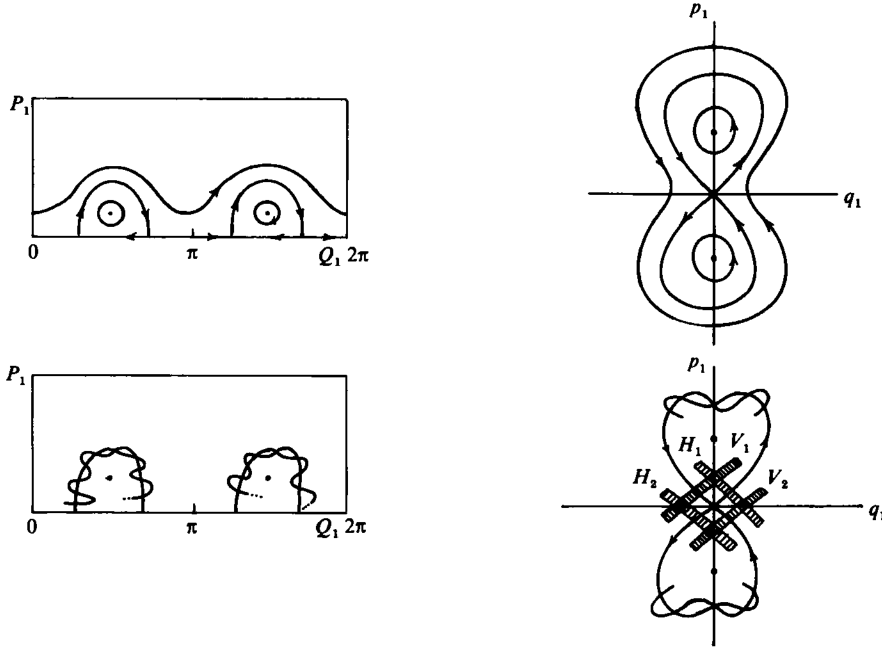

FIGURE 2. The invariant Cantor set in the original (p_1, q_1)-coordinate system.

action set $\{P_2 = P_2^*;\ P_k = 0,\ k = 3, \ldots, N\}$. Now the truncated system and the full system can be written in the forms (cf. (2.11)–(2.12)):

$$\dot{y} = \epsilon X^0(y) + \mu X^1(y), \tag{5.1a}$$

and

$$\left.\begin{aligned} \dot{y} &= \epsilon X^0(y) + \mu X^1(y) + \epsilon^2 X^2(y, \theta;\, \epsilon, \mu) + \epsilon\mu X^3(y, \theta, \epsilon, \mu), \\ \dot{\theta} &= \omega, \end{aligned}\right\} \tag{5.1b}$$

where X^2 and X^3 are 2π-periodic in θ and $X^{2,3}(y, \theta, 0, 0)$ are $O(1)$. Here $\theta = \omega t$ is a time-like coordinate and (5.1*b*) represents a *suspension* of the time-periodic vector field $\epsilon X^0 + \mu X^1 + \ldots$ in the extended phase space $\mathbb{R}^{2N} \times S^1$; cf. Guckenheimer & Holmes (1983, chap. 1.5).

To draw conclusions about the behaviour of (5.1*b*) from that of (5.1*a*), we consider a second Poincaré map, the period map $\Psi: \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ induced by the flow of (5.1*b*) on the global cross-section $\theta = 0$ ($t = 0, 2\pi/\omega, \ldots$). Since $\epsilon^2/\mu \ll 1$, this map is well approximated by the time $2\pi/\omega$ flow map $\phi_{2\pi/\omega}$ of (5.1*a*). (A simple Gronwall estimate (Hartman 1964) shows that $\| \Psi - \phi_{2\pi/\omega} \| = O(\epsilon^2/\mu)$.) Thus we can apply the normal hyperbolic persistence theory to the maps $\phi_{2\pi/\omega}$ and $\Psi = \phi_{2\pi/\omega} + O(\epsilon^2/\mu)$ and conclude that the period map Ψ has an invariant set Ω' near to the invariant set Ω of $\phi_{2\pi/\omega}$.

To prepare for the final section of this paper, we recall the canonical transformations

$$(q_n, p_n) \to (\hat{q}_n, \hat{p}_n) \to (Q_n, P_n) \tag{5.2}$$

of §2, and, inverting these transformations, we sketch the structure of the Cantor set Ω (or Ω') in the original (q_n, p_n) coordinates on the global cross-section $\theta = 0$ ($t = 0$). Since (P_n, Q_n) and $(\hat{p}_n, \hat{q}_n)$ play the roles of action and angle, or amplitude and phase, the double heteroclinic loop structure of the integrable system in (P_1, Q_1)-space (figure 1) becomes the characteristic figure-of-eight period-two resonance in the (p_1, q_1) slice of the cross-section $\theta = 0$; see figure 2, and note that the pair of elliptic points on the p_1 axis represents an orbit of period two for the $2\pi/\omega$-period map.

We remark that the Cantor set whose construction is outlined in the Appendix can be embedded in a manner which reflects the fact that orbits originating near the origin in (p_1, q_1)-space can describe loops with p_1 positive *or* negative in arbitrary sequences (cf. Greenspan & Holmes 1983, §10.6.2). Finally, we note that, since $\hat{p}_1 = P_1$, $\hat{p}_2 = \frac{1}{2}(P_1 - P_2)$ and $P_2 \approx P_2^*$ remains almost constant, as the envelope of $p_1 = -(2g\hat{p}_1/\omega_1)^{\frac{1}{2}} \sin \hat{q}_1$ increases that of $p_2 = -(2g\hat{p}_2/\omega_2)^{\frac{1}{2}} \sin \hat{q}_2$ decreases. We now turn to the physical implications of this for fluid-surface motions.

6. Conclusions and physical implications

In the original formulation of the finite-dimensional model (2.2), the configuration variables $q_n(t)$ represent amplitudes of 'linear' modes (eigenfunctions) $\psi_n(x)$ of the surface-displacement function $\eta(x, t)$; cf. (1.1) and Miles (1976, equation (2.4*a*)). We now use this representation to draw conclusions from the results established above.

We first note that, from proposition 4.1 and the remarks of §5, the amplitudes $\hat{p}_k = P_k$ of all modes but the first two remain small (of $O(\epsilon^2/\mu)$) owing to the presence of the damping terms $-2\mu\alpha_k P_k$ in (2.13). Thus, these higher modes respond at most weakly, as 'slave' modes, and the energy resides primarily in the first two modes. Of course these modes are not necessarily the lowest in frequency, and perhaps should rather be called *resonant* modes, for they are characterized by the frequency *relationship* (2.4) and not absolute frequency values. In particular, they need not have characteristically simple 'low-mode' spatial forms (cf. Ciliberto & Gollub 1985).

It is appropriate here to remark that damping can sometimes offset higher-order resonances of the form $\omega_k \pm \omega_l \pm \omega_m = O(\epsilon)$ (which we have assumed not to occur). For example, if the coefficients α_k increase sufficiently rapidly with k, then strong hyperbolic attraction due to $-2\mu\alpha_k P_k$ dominates the $O(\epsilon)$ resonant terms. Using this, it might be possible to deal with multiple resonances arising in 'harmonic-mode' problems such as the rectangular tank of Miles (1984*e*).

We now consider the dynamics exhibited by the two modes ψ_1 and ψ_2 as expressed by the time histories of q_1 and q_2. Recall that the properties of the Poincaré return map Φ induced by the flow of the truncated system (Theorem 4.2) were obtained from a computation involving orbits which continually return to the neighbourhood of $(Q_1, P_1) = (\frac{1}{2}\cos^{-1}(-\Omega/\beta(2P_2^*)^{\frac{1}{2}}), 0)$ but do not remain there too long, the characteristic recurrence time of such orbits being $O(K \ln(\epsilon/\mu))$. Since K can be chosen with some freedom, typical orbits associated with the invariant set Λ and its suspension Ω are quasi-periodic, and spend a relatively long time near $P_1 = 0$ ($|q_1| = 0$), with shorter excursions to higher values of P_1. The time dependence suppressed in the canonical coordinate changes and averaging procedure, causes a rapid oscillation at the frequencies $\frac{1}{2}\omega$ in q_1 and ω in q_2, on which the slowly varying envelope described above is superimposed ((2.6)). Finally, since $\hat{p}_2$ decreases when $\hat{p}_1$ increases ($P_2 = 2\hat{p}_2 + \hat{p}_1$ remaining approximately constant), we obtain the typical time histories of the modal amplitudes q_1 and q_2 sketched in figure 3. These time histories indicate an interaction in which energy is exchanged between the spatial modes $\psi_1(x)$ and $\psi_2(x)$ in an irregular fashion.

We conclude with some comments on the relevance of the foregoing analysis to experiments on surface waves such as those of the Keolian and Gollub groups, concentrating on the latter experiments, and especially those reported by Ciliberto & Gollub (1985), in which relatively detailed information on spatial structures is available. These experiments and our theory both exhibit a primary quasi-periodic

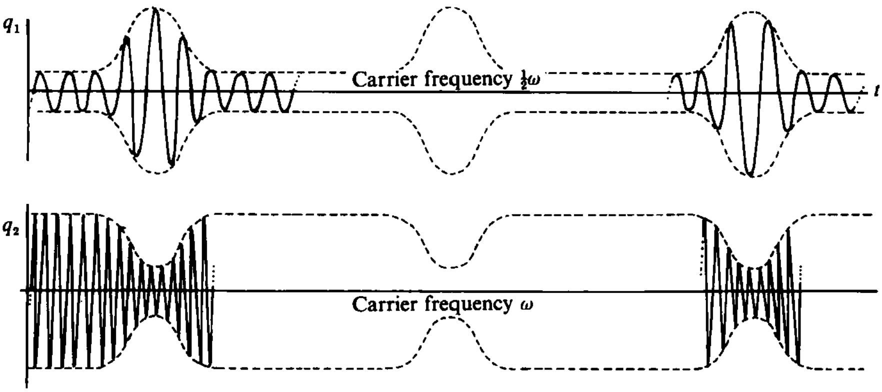

FIGURE 3. Typical time histories of modal amplitudes.

interaction between a pair of modes, with very little energy present in other modes. We remark that in Keolian & Rudnick (1984) there is some evidence of an interaction involving *three* modes, but that this is not as clear as the two-mode observations of Ciliberto & Gollub (1985).

A major difference between the present analysis and the Ciliberto–Gollub experiment is in the resonance condition. Following Miles (1976), we have assumed a 2:1:1 resonance, (2.4), whereas figure 3 of Ciliberto & Gollub (1985) suggests that their system is closer to a 2:2:1 resonance. Reference to (2.2) shows that 1:1 internal resonance terms do not emerge until *second*-order, in terms of the type $(\hat{p}_k \hat{p}_l \hat{p}_m \hat{p}_n)^{\frac{1}{2}} \cos \hat{q}_k \cos \hat{q}_l \sin \hat{q}_m \sin \hat{q}_n$, and thus that second-order averaging will be required for such a problem. We have not yet attempted this calculation. (We remark that Miles (1984*c, d*) develops the 1:1:1 resonance case with horizontal excitation, but, upon examination, the integrable limiting system (Miles 1984*c*, equations (3.12, 4.1)) is extremely degenerate and does not possess any homoclinic manifolds to hyperbolic invariant sets. Thus the perturbation theory developed here cannot be applied.) In spite of this discrepancy, there are clear qualitative similarities between the envelopes of $q_1(t)$ and $q_2(t)$ in figure 3 and the slowly varying modal amplitudes of figure 5 in Ciliberto & Gollub (1985). Those authors note that the typical modulation has a period of about 15 seconds compared to the driving oscillation of $\approx$ 16 Hz. While we cannot make numerical comparisons, this is consistent with our $O(K \ln(\epsilon/\mu))$ quasi-periodic modulation. Since the direct measurement of surface motion at a point in space, or of its time envelope, involves addition of the modes $q_1(t)\psi_1(x)$ and $q_2(t)\psi_2(x)$, it is less easy to compare other observations in the Ciliberto–Gollub work. However, we do note that, if the force amplitude $\mu\omega^2\omega_1/2g$ is small in comparison with damping terms $\mu\alpha_k$, then *no* homoclinic orbits survive, for the term (4.2) of the damped Melnikov function dominates and the function has no zeros. Thus, for given dissipative forces, there is a critical force level that must be exceeded before chaotic motions can occur. This is observed experimentally. We also note that our results rely on relatively close resonance ratios ($\epsilon\Delta$ and $\epsilon\Omega$ should be small quantities) and thus one only expects to find chaotic motions for sufficiently large force amplitudes and near specific forcing frequencies.

This research was partially supported by NSF CME 84-02069 and ONR under N00014-85-k-0172. The author would like to thank the referees for their suggestions and comments.

Appendix. Homoclinic orbits without reduction, and their consequences for Hamiltonian and weakly damped two-degree-of-freedom systems

As in (3.1), let

$$\left.\begin{aligned} H^\epsilon(q,p,I,\theta) &= H^0(q,p,I)+\epsilon H^1(q,p,I,\theta)+O(\epsilon^2),\\ (q,p)&\in\mathbb{R}^2,\quad (I,\theta)\in\mathbb{R}^+\times S^1, \end{aligned}\right\} \tag{A 1}$$

be a perturbed Hamiltonian energy function, with H^1 2π-periodic in θ. Suppose that, for each I in some interval $L\subset\mathbb{R}$, the unperturbed Hamiltonian $H^0(q,p,I)$ has a level set $(H^0)^{-1}(h)$ $(h=h(I))$ which contains a homoclinic orbit $(q,p)=\bar{x}(t,I)$ to a hyperbolic saddle point $(q,p)=\bar{x}_0(I)$. Let the corresponding angle variable on the homoclinic orbit based at $(\bar{x}(t,I),I,0)$ be

$$\bar{\theta}(t,I)=\int_0^t \frac{\partial H^0}{\partial I}(\bar{x}(s,I),I)\,\mathrm{d}s. \tag{A 2}$$

This implies that the unperturbed system has a normally hyperbolic two-manifold $\Gamma_0=\{(q,p,I,\theta)\mid(q,p)=\bar{x}_0(I),\,I\in L\}$ with three-dimensional stable and unstable manifolds $W^s(\Gamma_0)$, $W^u(\Gamma_0)$ which intersect in a three-manifold $\mathscr{M}=W^s(\Gamma_0)\cap W^u(\Gamma_0)$, parametrized by I, θ and t and given by

$$(q,p,I,\theta)=(\bar{x}(t,I),I,\bar{\theta}(t,I)+\theta), \tag{A 3}$$

see figure 4, below. Let $\Omega(q,p,I)=\partial H^0/\partial I$ be the unperturbed frequency. We shall *not* assume that $\Omega(\bar{x}(t,I),I)$ remains non-zero on $\mathscr{M}$, although ultimately we shall require that $\Omega(\bar{x}_0(I),I)\neq 0$ on Γ_0. Finally, let

$$\{H^0,H^1\}_{(q,p)}=\frac{\partial H^0}{\partial q}\frac{\partial H^1}{\partial p}-\frac{\partial H^0}{\partial p}\frac{\partial H^1}{\partial q}$$

denote the 'q–p' Poisson bracket of H^0 and H^1.

As in the usual Melnikov method, we will develop a perturbation theory to detect transverse homoclinic orbits to non-wandering sets near Γ_0. We start with a perturbation result which is a direct consequence of the theory of hyperbolic manifolds (Hirsch *et al.* 1977):

LEMMA A 1. *For $\epsilon\neq 0$ sufficiently small, the Hamiltonian system* (A 1) *possesses a normally hyperbolic, invariant two-manifold $\Gamma_\epsilon=\Gamma_0+O(\epsilon)$ with three-dimensional local stable and unstable manifolds $W^{s,u}_{loc}(\Gamma_\epsilon)$ ϵ-close to $W^{s,u}_{loc}(\Gamma_0)$.*

We will now sketch the proof of several results which permit us to prove the existence of chaotic motions in systems like (A 1).

If we assume that the perturbed action,

$$I_\epsilon=\int_0^t -\frac{\partial H^1}{\partial\theta}(q_\epsilon,p_\epsilon,I_\epsilon,\theta_\epsilon)\,\mathrm{d}t, \tag{A 4}$$

remains uniformly ϵ-close to the unperturbed action $I=I_0=$ constant; then, as in the standard Melnikov theory, we can write orbits in the stable and unstable manifolds as power series in ϵ, uniformly valid in the semi-infinite intervals indicated:

$$\left.\begin{aligned} &\text{stable:}\quad (x^s_\epsilon,I^s_\epsilon,\theta^s_\epsilon)=(\bar{x}(t,I)+\epsilon x^s_1(t,I),\ I+\epsilon I^s_1(t,I),\ \bar{\theta}(t,I)+\epsilon\theta^s_1(t,I))+O(\epsilon^2)\\ &\qquad\qquad (0\leqslant t<\infty);\\ &\text{unstable:}\quad (x^u_\epsilon,I^u_\epsilon,\theta^u_\epsilon)=(\bar{x}(t,I)+\epsilon x^u_1(t,I),\ I+\epsilon I^u_1(t,I),\ \bar{\theta}(t,I)+\epsilon\theta^u_1(t,I))+O(\epsilon^2)\\ &\qquad\qquad (-\infty<t\leqslant 0). \end{aligned}\right\} \tag{A 5}$$

To guarantee (A 4) it is sufficient (but not necessary) to assume that $\partial H^1/\partial\theta \to 0$ exponentially as $\bar{x}(t, I) \to \bar{x}_0(I)$. For the proof of uniform validity in such cases, see Guckenheimer & Holmes (1983, chap. 4.5).

To characterize the splitting of the perturbed manifolds $W^s(\Gamma_\epsilon)$, $W^u(\Gamma_\epsilon)$ in the four-dimensional phase space, we pick a point $(\bar{x}(0, I_0), I_0, \theta_0)$ on the unperturbed manifold $\mathscr{M}$ and construct the normal

$$n = \left(\frac{\partial H^0}{\partial q}(\bar{x}(0, I_0), I_0), \frac{\partial H^0}{\partial p}(\bar{x}(0, I_0), I_0), 0, 0\right)$$

to $\mathscr{M}$ in the slice $I = I_0$. The scalar quantity

$$M(I_0, \theta_0) = n(\bar{x}(0, I_0))\cdot[x^u_\epsilon(0, I_0, \theta_0) - x^s_\epsilon(0, I_0, \theta_0]) \tag{A 6}$$

is then a good measure of the splitting of stable and unstable manifolds for the perturbed problem. In fact $M/\|n\| = (M/((\partial H^0/\partial q)^2 + (\partial H^0/\partial p)^2)^{\frac{1}{2}})(\bar{x}(0, I), I))$ is the actual distance between the manifolds, measured along n. Here $x^{u,s}_\epsilon(0, I_0, \theta_0)$ represent solutions based at $I = I_0$, $\theta = \theta_0$, $x \approx \bar{x}(0, I_0)$ lying in the perturbed unstable and stable manifolds respectively. In view of Lemma A 1 and the remarks following it, such solutions do exist for ϵ sufficiently small.

Let us write the perturbed differential equation corresponding to the Hamiltonian (A 1) as

$$\left.\begin{aligned} \dot{x}_\epsilon &= f(x_\epsilon, I_\epsilon) + \epsilon g(x_\epsilon, I_\epsilon, \theta_\epsilon) + O(\epsilon^2), \\ \dot{I}_\epsilon &= \epsilon h(x_\epsilon, I_\epsilon, \theta_\epsilon) + O(\epsilon^2), \\ \dot{\theta}_\epsilon &= \Omega(x_\epsilon, I_\epsilon) + O(\epsilon), \end{aligned}\right\} \tag{A 7}$$

where $$f = \left(\frac{\partial H^0}{\partial p}, -\frac{\partial H^0}{\partial q}\right), \quad g = \left(\frac{\partial H^1}{\partial p}, -\frac{\partial H^1}{\partial q}\right), \quad h = \frac{-\partial H^1}{\partial \theta}, \quad \text{etc.}$$

Using this notation we can write

$$M(I_0, \theta_0) = f(\bar{x}(0, I_0), I_0) \wedge (x^u_\epsilon(0, I_0, \theta_0) - x^s_\epsilon(0, I_0, \theta_0)), \tag{A 8}$$

where the skew-symmetric wedge product is defined by $a \wedge b = a_1 b_2 - a_2 b_1$. The first variational equation corresponding to (A 7), evaluated on the homoclinic manifold $\mathscr{M}$, is

$$\left.\begin{aligned} \dot{x}^{s,u}_1 &= D_x f(\bar{x}(t, I_0), I_0)\, x^{s,u}_1 + D_I f(\bar{x}(t, I_0), I_0)\, I^{s,u}_1 + g(\bar{x}(t, I_0), I_0, \bar{\theta}(t, I_0) + \theta_0), \\ \dot{I}^{s,u}_1 &= h(\bar{x}(t, I_0), I_0, \bar{\theta}(t, I_0) + \theta_0). \\ \dot{\theta}^{s,u}_1 &= \dots . \end{aligned}\right\} \tag{A 9}$$

Using (A 9) and (A 5) in (A 8) we find

$$\begin{aligned} M(I_0, \theta_0) &= \epsilon f(\bar{x}(0, I_0), I_0) \wedge (x^u_1(0, I_0, \theta_0) - x^s_1(0, I_0, \theta_0)) + O(\epsilon^2) \\ &\overset{\text{def}}{=} \epsilon(\Delta^u(0, I_0, \theta_0) - \Delta^s(0, I_0, \theta_0)) + O(\epsilon^2). \end{aligned} \tag{A 10}$$

To compute $M(I_0, \theta_0)$, we let $\Delta^{u,s}$ vary with t (replace '0' by 't' in the definition) and differentiate. After a straightforward computation using the chain rule, the first variational equation (A 9), and the fact that f is Hamiltonian and so $\operatorname{tr} D_x f \equiv 0$, we find

$$\dot{\Delta}^u = (f \wedge g) + (f \wedge D_I f)\, I^u_1. \tag{A 11}$$

There is a similar expression for $\dot{\Delta}^{s}$. Integrating these expressions yields

$$\left.\begin{aligned}\Delta^{u}(-S,I_0,\theta_0) &= \Delta^{u}(0,I_0,\theta_0)+\int_0^{-S}[(f\wedge g)+(f\wedge D_I f)\,I_1^{u}]\,dt,\\ \Delta^{s}(T,I_0,\theta_0) &= \Delta^{s}(0,I_0,\theta_0)+\int_0^{T}[(f\wedge g)+(f\wedge D_I f)\,I_1^{s}]\,dt.\end{aligned}\right\}\tag{A 12}$$

Now since $\Delta^{u,s}=f\wedge x_1^{u,s}$, $x_1^{u,s}$ are bounded and $f=[\partial H^0/\partial p,\ -\partial H^0/\partial q]\to 0$ as $t\to\infty$ (we approach the hyperbolic saddle point in x-space), the quantities Δ^{u}, $\Delta^{s}\to 0$ as S, $T\to\infty$, and subtracting the components of (A 12) yields

$$\left.\begin{aligned}M(I_0,\theta_0) &= \epsilon\int_{-\infty}^{\infty}[f\wedge g+(f\wedge D_I f)\,I_1(t)]\,(\bar{x}(t,I_0),I_0,\bar{\theta}(t,I_0)+\theta_0)\,dt,\\ \text{where}\qquad I_1(t) &= I_1^{u,s}(t)=\int_0^{t}h(\bar{x}(s,I_0),I_0,\bar{\theta}(s,I_0)+\theta_0)\,ds.\end{aligned}\right\}\tag{A 13}$$

We next observe that, if $g=[\partial H^1/\partial q,\ -\partial H^1/\partial p]$ is a Hamiltonian perturbation, then $f\wedge g=\{H^0,H^1\}$ and

$$\begin{aligned}f\wedge D_I f &= -\frac{\partial H^0}{\partial p}\frac{\partial^2 H^0}{\partial I\,\partial q}+\frac{\partial H^0}{\partial q}\frac{\partial^2 H^0}{\partial I\,\partial p}\\ &= \left\{H^0,\frac{\partial H^0}{\partial I}\right\}=\{H^0,\Omega\};\end{aligned}\tag{A 14}$$

also, since $h=-\partial H^1/\partial\theta$, we can rewrite $I_1(t)$ as

$$I_1(t)=\int_0^{t}-\frac{\partial H^1}{\partial\theta}\cdot\left(\frac{dt}{d\theta}\right)d\theta=\int_0^{t}-\frac{\partial H^1}{\partial\theta}\cdot\frac{1}{\Omega}\,d\theta=\frac{-H^1}{\Omega},\tag{A 15}$$

provided that $\Omega\neq 0$. In this case, we have

$$\left.\begin{aligned}M(I_0,\theta_0) &= \epsilon\int_{-\infty}^{\infty}\left[\{H^0,H^1\}-\left\{H^0,\frac{H^1}{\Omega}\right\}\right](\bar{x}(t,I_0),I_0,\bar{\theta}(t,I_0)+\theta_0)\,dt\\ &= \int_{-\infty}^{\infty}\Omega\left\{H^0,\frac{H^1}{\Omega}\right\}(\bar{x}(t,I_0),I_0,\bar{\theta}(t,I_0)+\theta_0)\,dt.\end{aligned}\right\}\tag{A 16}$$

Apart from the presence of the additional factor of Ω (due to the different normalization of the distance measure based on $n=[\partial H^0/\partial q,\ \partial H^0/\partial p,0,0]$ of the non-reduced vector field rather than $n_R=(1/\Omega)[\partial H^0/\partial q,\ \partial H^0/\partial p]$ for the reduced field), the Melnikov function (A 16) is the same as that obtained in the reduction analysis, (3.8), cf. Holmes & Marsden (1983).

Now suppose that $M(I_0,\theta_0)$ has simple zeros as a function of θ_0 for fixed I_0. Use of the implicit-function theorem then implies that the actual distance $d=(\epsilon M+O(\epsilon^2))/\|n\|$ between the perturbed manifolds also has simple zeros as θ_0 varies. Thus, on each slice $I=I_0=$ constant, the perturbed manifolds appear as in figure 4(*b*).

It remains to determine behaviour in the I-direction, since $\dot{I}=O(\epsilon)\neq 0$ and I does not remain constant for the perturbed problem. However, $I=I_0+O(\epsilon)$ does remain close to its unperturbed value, and we will show that conservation of total energy H^{ϵ} together with the non-degeneracy assumption that the unperturbed frequency $\Omega(\bar{x}_0(I),I)\neq 0$ on Γ_0, guarantees that I returns to its initial value on Γ_{ϵ} as the

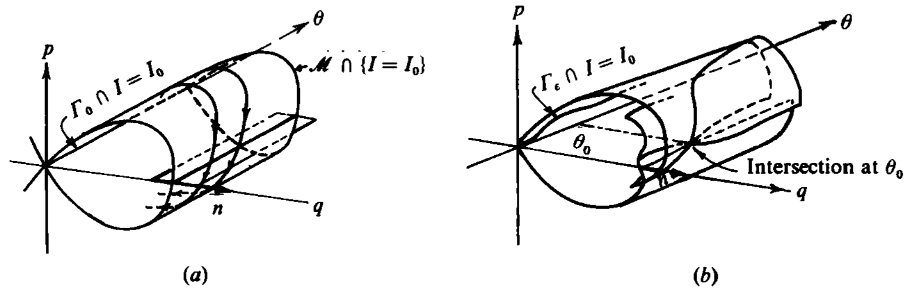

FIGURE 4. The homoclinic orbits and invariant manifolds in a slice I = constant. (*a*) $\epsilon = 0$, unperturbed; (*b*) $\epsilon \neq 0$, perturbed. $M(I_0, \theta_0)$ has simple zeros.

perturbed orbit returns to Γ_ϵ. This follows from the fact that each level set H^0 = constant intersects Γ_0 transversely at a (locally) unique action value I, since $\nabla H \cdot (0,0,1,0) = \Omega(\bar{x}_0(I), I) \neq 0$ on Γ_0, by assumption. This in turn implies that the level sets of H^ϵ intersect Γ_ϵ in unique action sets $I = I_\epsilon(\theta)$, which are, of course, periodic orbits of the flow. Thus any homoclinic orbit to Γ_ϵ is necessarily asymptotic to the *same* periodic orbit as $t \to -\infty$ and $t \to +\infty$. We summarize:

PROPOSITION A 2. *Suppose $M(I_0, \theta_0)$ has simple zeros as θ_0 varies for each fixed $I_0 \in L \subset \mathbb{R}$. Then, for $\epsilon \neq 0$ sufficiently small, there exists a one-parameter family of closed orbits $I = I_\epsilon(\theta, I_0)$ lying in the perturbed two-manifold Γ_ϵ each of whose stable and unstable manifolds intersect transversely. $M(I_0, \theta_0)$ is given by*

$$\left.\begin{aligned} M(I_0,\theta_0) &= \int_{-\infty}^{\infty} [\{H^0, H^1\} + \{H^0, \Omega\}\, I_1(t)]\,(\bar{x}(t, I_0), I_0, \bar{\theta}(t, I_0) + \theta_0)\,\mathrm{d}t; \\ I_1(t) &= \int_0^t \frac{-\partial H^1}{\partial \theta}\,(\bar{x}(s, I_0), I_0, \bar{\theta}(s, I_0) + \theta_0)\,\mathrm{d}s, \\ \bar{\theta}(t, I_0) &= \int_0^t \frac{\partial H^0}{\partial I}\,(\bar{x}(s, I_0), I_0)\,\mathrm{d}s, \end{aligned}\right\} \quad \text{(A 17)}$$

where $\bar{x}(t, I_0)$ is the unperturbed homoclinic orbit based at $\bar{x}_0(I_0)$ in the slice $I = I_0$.

We next consider the case of non-Hamiltonian perturbations. The proof of existence of the perturbed manifold Γ_ϵ and the characterization, via $M(I_0, \theta_0)$, of the splitting of its stable and unstable manifolds goes through just as before, although now M is given by (A 13) since g, h are not Hamiltonian. However, since total energy H^ϵ is not conserved, we must pay more careful attention to the slow 'drift' in the I-direction. First, however, arguing as in lemma A 1 and proposition A 2 above we have:

PROPOSITION A 3. *Suppose $M(I_0, \theta_0)$ given by* (A 13) *has simple zeros as θ_0 varies for each $I_0 \in L \subset \mathbb{R}$. Then, for $\epsilon \neq 0$ sufficiently small, there exists an invariant two-manifold Γ_ϵ ϵ-close to Γ_0 whose stable and unstable manifolds intersect transversely.*

Remarks. Since solutions can now escape from the neighbourhood of any level set I = constant in Γ_0, or, equivalently, $I_1(t)$ might not remain uniformly bounded, to prove the persistence of the invariant manifold Γ_ϵ we use a set of bump functions which cut off the drift due to the perturbation outside the neighbourhood of interest (cf. Hirsch *et al.* 1977; also see Robinson 1983).

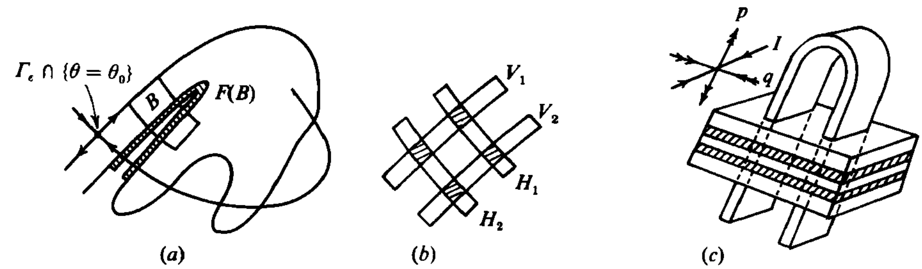

FIGURE 5. The horseshoe. (a) The horseshoe projected onto the (q,p)-plane; (b) strips; (c) the horseshoe in (q,p,I)-space; contraction in I case.

To compute drift in the I-direction we consider the integral

$$\Delta(I_0,\theta_0;S,T) = I(T)-I(-S) = \int_{-S}^{T} \dot{I}(t)\,\mathrm{d}t = \int_{-S}^{T} h(\hat{x}(t,I_0),I_0,\hat{\theta}(t,I_0)+\theta_0\,\mathrm{d}t, \quad \text{(A 18)}$$

evaluated either on the homoclinic manifold $\mathscr{M}$, in which case $\hat{x} = \bar{x}(t,I_0)$ and $\hat{\theta} = \bar{\theta}(t,I_0)$, or in its neighbourhood. If ΔI has a simple zero in I_0 at I_0^*, θ_0^* for a fixed time interval $(-S,T) \subseteq \mathbb{R}$, then application of the implicit-function theorem shows that there exists an orbit near $(\hat{x}(t,I_0^*), I_0^*, \hat{\theta}(t,I_0^*)+\theta_0^*)$ that returns to its initial-action value $I(-S)$ as $t \to T$.

We next use the homoclinic structure of proposition A 3 to show that, under suitable circumstances, this orbit is also recurrent in the variables $(q,p) = x$ and θ and hence that the 'chaotic', homoclinic behaviour of the Hamiltonian case (proposition A 2) persists near the action set $I_0 = I_0^*$ when damping is added.

We consider the three-dimensional Poincaré map induced by the flow of (A 7) on the cross-section $\theta = \theta_0$ near the perturbed manifold $\Gamma_\epsilon \cap \{\theta = \theta_0\}$. To ensure that solutions return to $\{\theta = \theta_0\}$, we require that $\Omega(q,p,I_0) \neq 0$ on Γ_ϵ. The existence of transverse homoclinic orbits to Γ_ϵ implies that, in the (q,p)-coordinate plane, a small region B, sufficiently close to the 1-manifold $\mathscr{M}$, is carried around near $\mathscr{M}$ and returns near $\Gamma_\epsilon \cap \{\theta = \theta_0\}$ bent in the horseshoe shape of figure 5. Thus, as in Smale (1963) or Moser (1973), two strips H_1, H_2 can be found whose images V_1, V_2 intersect H_1, H_2 in such a fashion that the (projected) return map $F: B \to \mathbb{R}^2$, restricted to

$$\bigcup_{i,j=1}^{2} \left(\bigcap_{i,j=1}^{2} H_i \cap V_j\right),$$

is a bounded perturbation of a (piecewise) linear map with real eigenvalues $|\lambda_1| < 1 < |\lambda_2|$. Then, in the two-dimensional case, one can prove the existence of an invariant Cantor set $\Lambda = \bigcap_{n=-\infty}^{\infty} F^n(B)$ on which F is conjugate to a shift map on two symbols. This is Smale's horseshoe, which displays the chaotic behaviour described in Guckenheimer & Holmes (1983, chap. 5).

The third (I-)direction does not really change this picture, provided that one has hyperbolic contraction (or expansion) in that direction. The Cantor-set construction goes through essentially unchanged, but with intersecting, 3-dimensional boxes H_i, V_i, instead of strips. (In fact the construction generalizes to infinite-dimensional maps: (cf. Holmes & Marsden 1981 and Hale & Lin 1984.) From the preceding discussion, the quantity ΔI provides the requisite measure of behaviour in the I-direction, and all that remains to be done is to estimate the time required for points in the box B (strips H_i) to flow around near $\mathscr{M}$ and return to B. This will give the time of flight $S+T$ required to compute ΔI.

Holmes & Marsden (1982*a*) have solved this time-of-flight problem. Basically, one needs to ensure that the preimage and image of a given transverse homoclinic point (say near a simple zero (I_0^*, θ_0^*) of $M(I_0, \theta_0)$) are chosen such that the tangent spaces of the stable and unstable manifolds subtend an angle of $O(1)$ at these points. Since the angle is $O[\epsilon(\partial M/\partial\theta)(I_0, \theta_0)] = O(\epsilon M)$ at (I_0^*, θ_0^*), one must let the expansion and contraction near the saddle-type set $\Gamma_0 \cap \{\theta = \theta_0\}$ act to increase ϵM to $O(1)$. This is easily estimated from the linearized map at $\Gamma_0 \cap \{\theta = \theta_0\}$ and one obtains $S, T = O(\ln(1/\epsilon))$. The final outcome of the arguments sketched above is then:

PROPOSITION A 4. *Suppose that the hypotheses of proposition* A 3 *are satisfied and, in addition, for* $S = T = K \ln(1/\epsilon)$ *fixed for some* K, $\Delta I(I_0, \theta_0, -T, T)$ *of* (A 18) *has a simple zero* I_0^* *in* I_0 *for fixed* $\theta_0 \in [0, 2\pi)$. *Then there exists an invariant Cantor set* Λ *near the point* $\Gamma_\epsilon \cap \{\theta = \theta_0\} \cap \{I = I_0^*\}$ *on which the flow of* (A 7) *induces a map conjugate to Smale's horseshoe map.*

COROLLARY A 5. *Λ contains countably many periodic orbits, including orbits of arbitrarily high period, uncountably many, bounded, non-periodic orbits, and a dense orbit. All the orbits are of 'saddle' type; thus Λ is a chaotic invariant set, but not an attractor.*

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